CARBONATED WATER

Suppose you have a regular 1-liter factory sealed bottle of carbonated water. You have turned the bottle cap slightly to unscrew it (so a hissing sound was heard) and screw the cap tightly again. Now you would observe bubbles of carbon dioxide (CO_2) rising upwards – large, at first, and then smaller ones. Let us study the process of bubble surfacing.

One can easily see that the shape of a small bubble is much closer to spherical than that of a bigger one.

1) Estimate the size of an immobile bubble such that the bubble shape approximates sphere with an accuracy of 10% or better. Water density is $\rho \approx 1$ g/cm^3, the surface tension $\sigma \approx 0{,}07$ N/m, and the free fall acceleration is taken to be $g \approx 9{,}8$ m/s^2. The numerical answer should be in mm.

Consider a bubble so small that it can be regarded as almost spherical. For instance, let the bubble initial diameter near the bottle bottom be $d_0 = 0{,}3\,\text{mm}$.

2) Figure out the bubble acceleration right after it has detached from the bottom. The CO_2 density inside the bubble at the given water temperature is $\rho_v \approx 0{,}002$ g/cm^3. The numerical answer should be in m/s^2.

Drag force exerted on a bubble, when it is moving in water, is linearly proportional to its cross-sectional area, water density, and the bubble velocity squared. We assume the proportionality factor for this problem to be $\beta = 0{,}2$.

3) Suppose that the bubble volume remains constant. What is the terminal velocity the bubble can reach? Write down the equation and evaluate the numerical value (in m/s).
4) Estimate the time it takes the bubble to reach the terminal velocity after detachment, i.e. when its acceleration becomes much less than g. Write down the equation and evaluate the numerical value (in seconds).

Actually, the density of CO_2 molecules dissolved in a liquid is much higher than that in a gas bubble. Therefore, the dissolved gas diffuses into the bubble and its radius grows. It is reasonable to assume that the growth rate of the bubble volume due to the diffusion is proportional to the bubble surface area and to an excess of the dissolved gas density and inversely proportional to the thickness of the liquid layer through which the gas diffuses (called the «depletion layer»). The faster bubble is moving the thinner is the effective depletion layer (due to circulation of water surrounding the bubble). A kinetic theory calculation gives for the depletion layer thickness: $\delta = const \cdot \sqrt{\frac{d}{v}}$, where v is the bubble velocity and d is its diameter.

5) Suppose it takes time T for a bubble to surface. Figure out the bubble diameter as a function of time t, providing it has increased by a factor $k = 3^{4/5} \approx 2{,}4$ during the surfacing. The answer should be an equation for $d(t)$ in terms of T, t, d_0, and $k = 3^{4/5}$. Neglect a change in the density of CO_2 dissolved in water during the bubble surfacing.
6) Determine time dependence $v(t)$ of the bubble velocity (the answer should include a formula for $v(t)$ expressed in terms of the parameters listed above and the terminal velocity v_0).

7) Determine the law of motion of a bubble, i.e. time dependence of its elevation h above the bottom (the answer should include a formula for $h(t)$ in terms of T, t, and v_0; take the same value $k = 3^{4/5}$).
8) Suppose the height of water column, which the bubble traverses on its way upward, equals $H = 30$ cm. Evaluate the time of bubble surfacing (the answer should be the explicit formula including the parameters given in the problem and calculate the numerical value in seconds).

During bubble surfacing *some heat is being released* (the drag force does a work by increasing the kinetic energy of turbulent flow which eventually dissipates as heat) and at the same time *some heat is being absorbed* (due to CO_2 evaporation from water into a bubble).

9) Assume that all the heat *released* during the surfacing is converted into heating the «column» of water which cross-section equals the average cross-section of a rising bubble. Using this assumption estimate (by the order of magnitude) the temperature increment of water in the «column». The specific heat capacity of water is $c_w \approx 4200$ J/kg·K. The answer should be given in Kelvin.
10) Evaporation into bubbles of 1 mole of CO_2 dissolved in water requires approximately 20 kJ of energy. Estimate the cooling effect by the order of magnitude and compare to the heating effect (see 9)). The answer should be given in Kelvin. What will the net result be? The answer should be either «+» (the temperature increases) or «–» (the temperature decreases). Assume that the pressure and temperature in the bottle change slightly and remain close to $p_0 \approx 120$ kPa and $T_0 \approx 290$ K.

PROPOSED SOLUTION AND ANSWERS

1) Bubble «oblateness» is caused by the difference of hydrostatic pressure above and below the bubble, while surface tension tries to keep the bubble shape spherical. Therefore, one obtains a reasonable estimate by assuming that the bubble shape remains spherical at the given accuracy if the force of surface tension exceeds the net force of hydrostatic pressure (i.e. buoyancy force for the bubble at rest) by one order of magnitude (i.e. approximately 10 times): $\sigma \cdot \pi d \geq 10\rho g \frac{\pi d^3}{6}$. Hence $d \leq \sqrt{\frac{3\sigma}{5\rho g}} \approx 2$ mm, so $d_{\max} \approx 2$ mm.

2) There is no drag force exerted on the bubble at zero velocity. Only the gravity and buoyancy forces act on it. However, using a «common» expression for the buoyancy force gives the absurd result: the acceleration of gas bubble must be about $500g$! Actually, at a non-zero acceleration the net force of hydrostatic pressure differs from the «static» value of buoyancy force. The simplest way to understand this is to take into account that bubble rising is equivalent to going down of the displaced amount of water due to gravity force. Obviously, the corresponding acceleration cannot exceed g even if there is no drag. At the start, the bubble acceleration is close to the maximum, i.e. $a_0 \approx g$.

Note: an accurate evaluation of the acceleration of a body of density ρ_v , which is surfacing in a liquid of density ρ, yields: $a = \frac{\rho - \rho_v}{\rho + \rho_v} g$. This indeed tends to g when $\rho >> \rho_v$.

3) At the stationary regime, when the bubble velocity equals to its terminal value v_0, the drag balances buoyancy force (which at zero acceleration equals its «static» value), and the gravity force exerted on the bubble can be neglected. Therefore, $\rho g \frac{\pi d_0^3}{6} = \beta \rho \frac{\pi d_0^2}{4} v_0^2$ and $v_0 = \sqrt{\frac{2 g d_0}{3\beta}} \approx 0{,}1$ m/s.

4) The bubble acceleration decreases from g to zero while its speed grows from zero to v_0. Therefore, it is reasonable to estimate that the bubble reaches the terminal velocity in a time $\tau \approx \frac{v_0}{g/2} \approx 2\sqrt{\frac{2d_0}{3g\beta}} \approx 0{,}02$ s. Notice that the height of water in the bottle significantly exceeds the distance traveled during this time (about 1 mm), so practically during the whole of surfacing the bubble is going up at the terminal velocity.

5) According to the previous estimate the bubble is going at the terminal velocity almost all the time during the surfacing, i.e. its velocity and the diameter are related as $v(y) = \sqrt{\frac{2gy}{3\beta}}$ (where y is the bubble diameter at a particular time). According to the problem statement, the rate of the bubble volume change is $\frac{d}{dt}\left(\frac{\pi y^3}{6}\right) = A \cdot \pi y^2 \cdot \frac{\Delta n}{\delta}$ (where Δn is the difference between CO_2 molecules density in water and inside the bubble and A is a numerical constant). Differentiating this equation one obtains: $\frac{dy}{dt} = \frac{2A\Delta n}{\delta}$. Now, using the equation given in 5) for the thickness of depletion layer and

recalling that approximately $\Delta n = const$, one obtains: $\frac{dy}{dt} = B\sqrt{\frac{v}{y}} = C \cdot y^{-1/4}$ (where B and C are new constants). Then $\int_{d_0}^{d(t)} y^{1/4} dy = C\int_0^t dt = Ct$, which gives after integration $\frac{4}{5}[(d(t))^{5/4} - d_0^{5/4}] = Ct \Rightarrow d(t) = \left\{d_0^{5/4} + \frac{5C}{4}t\right\}^{4/5}$. Since $d(T) = D = k \cdot d_0$ one can determine the constant: $d(t) = \left\{d_0^{5/4} + [D^{5/4} - d_0^{5/4}]\frac{t}{T}\right\}^{4/5} = d_0 \cdot \left\{1 + [k^{5/4} - 1]\frac{t}{T}\right\}^{4/5}$. Substituting $k = 3^{4/5}$, one gets finally: $d(t) = d_0 \cdot \left\{1 + 2\frac{t}{T}\right\}^{4/5}$.

6) Using the relation between velocity and diameter one obtains: $v(t) = \sqrt{\frac{2gd(t)}{3\beta}}$ or $v(t) \approx v_0 \cdot \left\{1 + [k^{5/4} - 1]\frac{t}{T}\right\}^{2/5} = v_0 \cdot \left\{1 + 2\frac{t}{T}\right\}^{2/5}$.

7) To determine the law of motion one should integrate the velocity: the elevation above the bottom is $h(t) = \int_0^t v(t)dt$. Hence, $h(t) \approx \frac{5v_0T}{14} \cdot \left\{\left[1 + 2\frac{t}{T}\right]^{7/5} - 1\right\}$.

8) Now we substitute $t = T$ in the derived equation and obtain $H \approx \frac{5(3^{7/5} - 1)v_0T}{14}$. Therefore $T \approx \frac{7}{5(3^{7/5} - 1)} H\sqrt{\frac{6\beta}{gd_0}} \approx 2{,}3$ s. The experimental value is a bit greater because the bubble at the end of surfacing is bigger and therefore more oblate, which leads to increasing drag force.

9) The heat released is equal (up to the sign) to the work done by drag force, so it can be estimated as $Q_+ \approx \beta \cdot \rho S_{\text{av}} \cdot v_{\text{av}}^2 \cdot H$ (the subscript «av» stands for the average value of a physical quantity). The mass of the water «column» described in the problem is $m \approx \rho S_{\text{av}} H$. Then the temperature increment of the «column» is estimated as $\Delta T_+ \approx \frac{\beta v_{\text{av}}^2}{c_{\text{w}}} \sim \frac{2gd_{\text{av}}}{c_{\text{w}}} \sim 10^{-6}$ K.

10) The absorbed heat is proportional to the amount of evaporated gas ν: $Q_- \approx r \cdot \nu$, where $r \approx 2 \cdot 10^4$ J/mole. Since $\nu \approx \frac{p_0}{RT_0}\frac{4\pi}{3}(k^3 - 1)d_0^3 \approx 7 \cdot 10^{-8}$ mole, $\Delta T_- \approx -\frac{r\nu}{c_{\text{w}}m} \approx -\frac{r\nu}{c_{\text{w}}\rho S_{\text{av}} H} \sim -10^{-3}$ K. Therefore, the net effect is, obviously, cooling. Notice that surfacing of a single bubble has almost no effect on the water temperature in the bottle.

TABLE OF ANSWERS

№	**Answer**	**Maximum score**
1	$d_{\text{max}} \approx 2$ mm. An answer deviating from this value by no more than 1mm is accepted.	**1**

2	A numerical answer must agree with the free fall acceleration by order of magnitude. $a_0 \approx g = 9{,}8\ \text{m/s}^2$.	1 + 1 = **2**
3	$v_0 = \sqrt{\dfrac{2gd_0}{3\beta}} \approx 0{,}1\ \text{m/s}$.	1(equation) + 1 (number) = **2**
4	$\tau \approx \dfrac{v_0}{g/2} \approx 2\sqrt{\dfrac{2d_0}{3g\beta}} \approx 0{,}02\ \text{s}$.	1(equation) + 1 (number) = **2**
5	$d(t) = d_0 \cdot \left\{1 + 2\dfrac{t}{T}\right\}^{4/5}$.	**4**
6	$v(t) \approx v_0 \cdot \left\{1 + 2\dfrac{t}{T}\right\}^{2/5}$, where $v_0 = \sqrt{\dfrac{2gd_0}{3\beta}}$.	**2**
7	$h(t) \approx \dfrac{5v_0T}{14} \cdot \left\{\left[1 + 2\dfrac{t}{T}\right]^{7/5} - 1\right\}$.	**3**
8	$T \approx \dfrac{7}{5(3^{7/5} - 1)} H \sqrt{\dfrac{6\beta}{gd_0}} \approx 2{,}3\ \text{s}$.	3 (equation) + 1 (number) = **4**
9	$\Delta T_+ \sim \dfrac{2gd_{av}}{c_w} \sim 10^{-6}\ \text{K}$.	**2**
10	$\Delta T_- \sim 10^{-3}\ \text{K}$; «–» (the net effect is cooling).	2 + 1 = **3**
Total		**25**

Problem 2: RADIOGRAPHY

Industrial radiography is one of the basic modern methods of materials science. A studied object is placed in a coherent monochromatic beam of X-ray photons of high intensity which scattering pattern is then analyzed. Sometimes *methods of spectrometry* are used, i.e. variation of intensity of the beam passing through a material is measured as a function of radiation wavelength. However, the most common methods of study of atomic structure are *diffraction methods*. They are based on analysis of the diffraction pattern resulting from elastic scattering of X-rays by atoms of the sample. Notice that radiation wavelength remains constant in elastic scattering. A common source of coherent X-rays of high intensity with a wide wavelength spectrum is *synchrotron* – a large ring storage of charged particles traveling at a speed close to the speed of light. Such particles are called *relativistic* because their motion is no longer described by Newtonian laws of classical mechanics, instead one must use the special theory of relativity (STR). The operating principle of synchrotron is based on the fact that a charged particle emits electromagnetic radiation at trajectory turns. Direction of particle velocity is changed by special bending magnets. A particle (e.g. electron) path in the synchrotron ring consists of straight segments, where electrons receive kinetic energy, and segments of almost constant curvature in a strong magnetic field of bending magnets. In the curved segments electron motion is highly accelerated, so they emit electromagnetic radiation in the X-ray range. A powerful synchrotron is operating at the Kurchatov Institute in Moscow.

Relativistic equation of motion of a charged particle in magnetic field is $\frac{d\vec{p}}{dt} = \overrightarrow{F_L} = q[\vec{v} \times \vec{B}]$, where the particle momentum $\vec{p} = \frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}} \equiv m\vec{v}$. Here c is the speed of light in vacuum ($c \approx 3 \cdot 10^8 m/s$), the quantity m_0 is the particle *invariant mass* and $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$ is the particle *relativistic mass*. The energy of relativistic particle $E = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} = mc^2$. Usually the energy of a microparticle is measured in electronvolts: 1 eV ≈ $(1{,}6 \cdot 10^{-19}$ C)·(1 V). The factor $\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}}$ is called the «*Lorentz factor*». If the particle speed is close to c, $\gamma \gg 1$.

1) Evaluate the Lorentz factor for an electron of energy $E_e = 2{,}5$ GeV (the invariant mass is $m_0 \approx 9 \cdot 10^{-31}$ kg, *the rest energy* is $m_0c^2 \approx 0{,}5$ MeV). By how many percent is the speed of such an electron less than c? The electron charge is $e \approx 1{,}6 \cdot 10^{-19}$ C. The answer should include formulae and numerical values.

2) Determine the curvature radius of electron trajectory in the field of bending magnet if electron energy in the synchrotron storage ring is maintained at $E_e = 2{,}5$ GeV and the induction of the field of bending magnet is $B = 1{,}7$ T. Electrons are traveling in a plane perpendicular to the magnetic field lines. The answer should include the formula and the numerical value.

Any accelerating charged particle emits electromagnetic radiation. The important feature of synchrotron radiation (i.e. radiation of relativistic particles with $\gamma \gg 1$ traveling along a curved path) is its «searchlight» nature: almost all the energy is radiated «forward» along the particle velocity in a narrow cone with half an aperture of $\varphi \approx \frac{1}{\gamma}$ (see Fig.1).

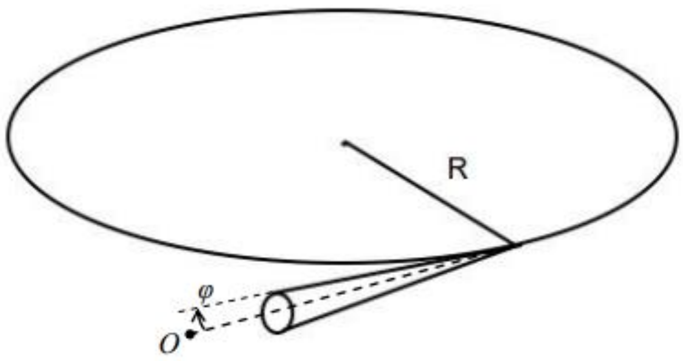


Fig. 1.

3) Suppose an «observer» O resides at the circular orbit plane and can be regarded as a point. In this case she detects radiation flashes corresponding to the short periods when she is inside the «searchlight» cone of orbiting particle. Determine the length Δl of the arc traversed by electron on the orbit when its radiation is detected by the observer. The electron energy and the magnetic induction are given in 2). The answer should be the equation.

4) Determine duration T_{sr} of the radiation «flash» detected by the observer. It must be taken into account that due to electron relativistic motion it takes different time for the «initial» and «final» portions of the «flash» to reach the observer. The answer should include the equation and the numerical value.

Thus, radiation of relativistic particle traveling along a circular path is observed as a bright short flash. The flash spectrum (frequencies and wavelengths) turns out to be very wide: the width of the frequency range corresponds to «characteristic» (or «synchrotron») frequency $\omega_{sr} \approx \frac{2\pi}{T_{sr}}$. A wide range of wavelengths provides a lot of opportunities for using synchrotron radiation in radiography. The characteristic wavelength of a particular source is an important quantity for practical applications.

5) Determine the characteristic wavelength of the source described in 2). The answer must be the equation and the numerical value.

The main method of deciphering the structure of a crystal material is *X-ray diffraction*. The radiation is being diffracted (elastically scattered) by atoms of a sample. The crystal serves as a *diffraction grid* for X-ray beam because its wavelength is of the same order of magnitude as a spacing between atomic planes. When the radiation is incident on the crystal at some angle, the reflected radiation is detected not only in the direction determined be the laws of geometrical optics but also at the angles for which the waves reflected by adjacent planes have optical path difference equal to an integer of radiation wavelength. These reflected waves mutually amplify at a remote detector resulting in a significant rise of intensity in the corresponding direction (*diffraction maximum*).

6) Using the condition of diffraction maximum derive the explicit formula for the direction at a diffraction maximum of X-rays reflected by a crystal which lattice consists of a single set of parallel planes. The beam is incident at the angle θ to the planes, the interplane spacing equals d (see Fig.2).

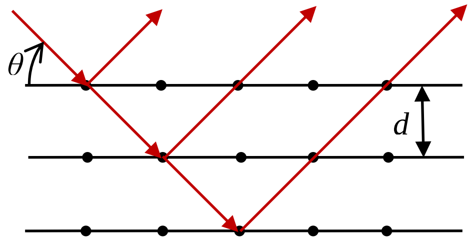


Fig. 2.

In a real crystal structure it possible to introduce different sets of equidistant parallel planes. Such a set can be defined by a vector perpendicular to the planes while directions at the diffraction maxima defined by the condition derived in 6) can be specified by diffraction angle 2θ (the angle between the incident and diffraction beams). In what follows the observed maximum of a given order for a specific set of parallel planes is called «*reflex*». Any atomic plane necessarily passes through the nodes of crystal lattice, so coordinates of the vector perpendicular to a particular set of parallel planes can be given by integers $\vec{K} = (h, k, l)$ providing the coordinate axes are aligned with the lattice principal axes (edges) and a distance is measured in lattice constants. Thus, a reflex can be determined by a set of three integers (hkl).

Superconductor is a very interesting object for the modern materials science. One of the most common and widely used low temperature superconductor is triniobium-tin Nb_3Sn. For instance, this superconductor is used in electrical circuits of the Large Hadron Collider. A unit cell of its lattice structure (i.e. a cell which repetitive translation along principal axes reproduces the whole crystal) is a cube of side L = 5,29 Å (1 Å = 10^{-8} cm).

7) Find the diffraction angle (the angle between the incident and diffracted beams) for reflex (110) at the first order of diffraction using the value of characteristic wavelength calculated above. The answer should be the formula and the numerical value.

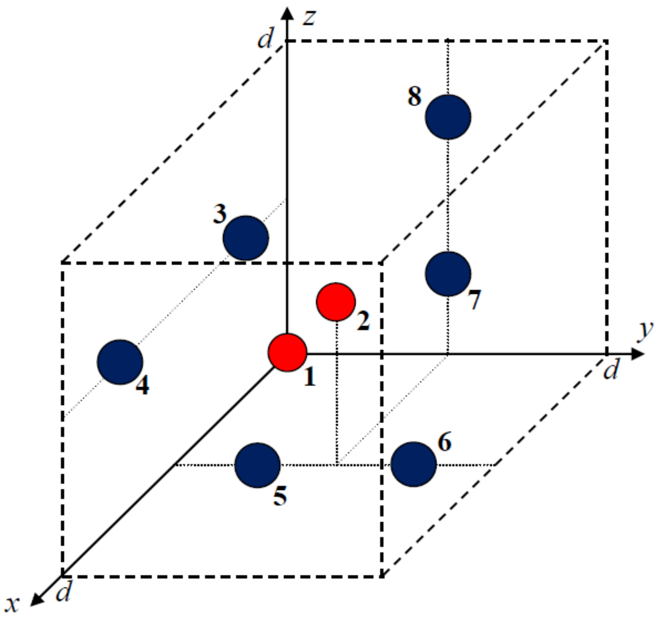


Fig. 3.

На рисунке надо заменить *d* на *L*.

In the Cartesian frame, which coordinate axes are aligned with the edges of crystal lattice, the coordinates of Sn atoms (in the units of *d*) are: (0; 0; 0), (0,5; 0,5; 0,5), and the coordinates of Nb atoms are: (0,25; 0; 0,5), (0,75; 0; 0,5), (0,5; 0,25; 0), (0,5; 0,75; 0), (0; 0,5; 0,25), (0; 0,5; 0,75), (see Fig.3). Atoms in the unit cell are numbered from 1 to 8 in the order they are listed in the text. A pattern of X-ray scattering is determined by the distribution of electrons in a crystal lattice (electrons are so much lighter than atomic nuclei, so they respond much stronger to the electromagnetic field of incident wave), i.e. by distribution of atoms of various elements. The ability of an isolated atom of a certain element A to scatter radiation is specified by a quantity $f(A)$ called *atomic scattering factor*. This quantity specifies the difference of wave scattering by electronic shell of a given atom compared to that one by free electrons. Atomic scattering factor is a complex quantity (i.e. $f = \alpha + i\beta$, where $i \equiv \sqrt{-1}$ is imaginary unit) and its absolute value squared $|f|^2 = \alpha^2 + \beta^2$ determines intensity of scattered radiation of an isolated atom. The intensity of an observed diffraction peak for a crystal lattice is calculated as the squared absolute value of the *reflex structure factor* $I \approx |F|^2$, which in turn is evaluated as:

$$F = \sum_n a_n(A) \cdot f(A) \cdot e^{2\pi i(hx+ky+lz)},$$

where sum is over all atomic positions in the unit cell. Here (x, y, z) are coordinates of atom at the n-th position; $f(A)$ is the atomic scattering factor of element A which atom resides at the n-th position, and $a_n(A)$ is *occupation* of the position by the element. Occupation of a position is the average (over the whole lattice) number of atoms of a certain element at the position. In ideal crystal (i.e. when all atoms of crystal lattice reside at their nodes) each position is occupied exactly by a single atom of a given element and there are no «extra» atoms, i.e. an occupation is either 1 or 0. For example, in the unit cell of Nb_3Sn at the 1-st position: $a_1(\text{Sn}) = 1$ and $a_1(\text{Nb}) = 0$. However, a real crystal structure has defects distorting ideal lattice, so occupations can be different. One of the most common defect of this sort is the so-called *antinode disordering* when atoms switch their positions. For instance, if in some cells atoms of Sn at position 1 switch to position 3, and atoms Nb in these cells switch from 3 to 1, occupation $a_1(\text{Sn})$ becomes less than 1 and $a_1(\text{Nb})$ becomes non-zero. Nevertheless, the gross occupation of any position remains equal to 1, i.e. any atom leaving its position switches to position of another atom and vice versa.

8) Suppose there is an antinode disordering in the considered structure, so the occupation of Nb positions by atoms of Sn becomes equal to δ, i.e.
$$a_3(\text{Sn}) = a_4(\text{Sn}) = a_5(\text{Sn}) = a_6(\text{Sn}) = a_7(\text{Sn}) = a_8(\text{Sn}) = \delta.$$
Determine other occupations $a_n(A)$ in the structure. Occupations of atoms of any element at positions $n = 1$ and $n = 2$ are the same, as it is the case for positions $n = 3, \ldots, 8$. Express the occupations via δ.
9) Assuming $f(\text{Nb}) \equiv f_{\text{I}}$ and $f(\text{Sn}) \equiv f_{\text{II}}$ to be known evaluate structure factor of reflex (110), using the occupations calculated in 8). The answer should be given as equation.

Hint: according to Euler's formula complex exponential is evaluated as $e^{i\varphi} = cos\varphi + i \cdot sin\varphi$.

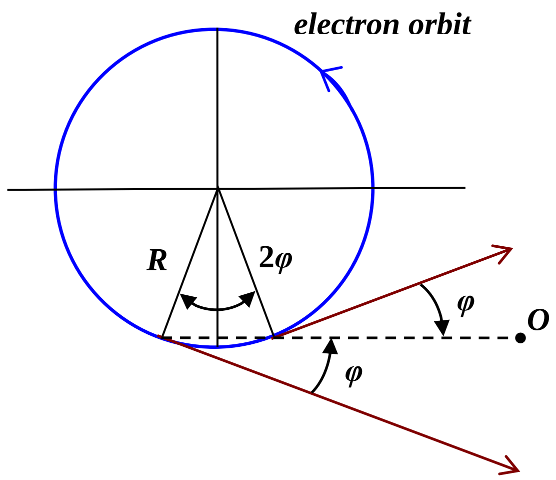
10) Determine the condition of quenching the reflex (110) (when its intensity vanishes) for the same structure. The answer should be given as a numerical value of δ.

PROPOSED SOLUTION AND ANSWERS

1) According to the given equations $\gamma = \frac{E_e}{m_0c^2} \approx \frac{2500}{0,5} = 5000$, i.e. $\gamma \gg 1$. Using the definition of the Lorentz factor one finds: $\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}$, whence $\frac{c-v}{c} \approx \frac{1}{2\gamma^2}$. Therefore, the speed of electrons in this machine is less than the speed of light by approximately 0,000002%.
2) In the field of bending magnet electrons are subjected to the Lorentz force which is perpendicular to particle velocity and does no work. Therefore, the absolute value of electron velocity (and the momentum) remains constant. The Lorentz factor remains constant as well and the equations of motion coincide with the corresponding equations of Newtonian mechanics in which mass is replaced with relativistic mass $m = \gamma m_0$. Hence, the curvature radius of electron path (called «Larmor radius») can be found from the equation $\frac{\gamma m_0 v^2}{R} = evB$. Since the electron

speed is practically the same as the speed of light one obtains: $R \approx \frac{E_e}{ecB} = \frac{\gamma m_0 c}{eB}$, or $R[m] = \frac{E[GeV]\cdot 10^9 \cdot 1{,}6\cdot 10^{-19}\left[\frac{J}{eV}\right]}{3\cdot 10^8\left[\frac{m}{s}\right]\cdot 1{,}6\cdot 10^{-19}[C]\cdot B[T]} \approx 4{,}9$ m.

3) If the «observer» of X-ray beam resides at the orbital plane, the detected radiation is produced by electrons traveling along the arc subtending the angle equal to the beam aperture (the cone aperture), see the figure. The corresponding arc length $\Delta l = R \cdot 2\varphi \approx \frac{2R}{\gamma}$. Equivalent answers are $\Delta l \approx \frac{2E_e}{ec\gamma B}$ and $\Delta l \approx \frac{2m_0 c}{eB}$.

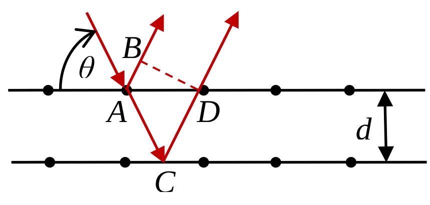


4) The time required for an electron to cover a distance Δl is $T = \frac{\Delta l}{v} \approx \frac{2R}{\gamma c}$, however, the duration of the flash is less due to the finiteness of the speed of light. To understand what is going on in detail, let us introduce the detector reference frame and consider two events: 1) emission of light by the electron at the beginning and 2) at the end of a flash. Let $t_1 = 0$ in the detector frame. Let the axis x be directed from the detector to the electron (since φ is small we can think of electron going along this axis during all the flash). Let electron coordinate (in electron frame) at $t_1 = 0$ be $x_1 = r$. The event 2 in electron frame then happens at $t_2 = T$ at the point $x_2 = r - vt_2$ (the electron approaches the detector). Now consider events: 3) detection of the beginning and 4) the end of flash by the detector. Obviously, the corresponding times are $t_3 = \frac{r}{c}$ и $t_4 = t_2 + \frac{r - vt_2}{c} = \frac{r}{c} + T\left(1 - \frac{v}{c}\right)$. Thus, the flash duration registered by the detector is $T_{sr} = t_4 - t_3 = T\left(1 - \frac{v}{c}\right) = \frac{R}{c\gamma^3} \approx 13{,}06 \cdot 10^{-20}$ s.

5) According to the above analysis the corresponding characteristic frequency is $\omega_{sr} = \frac{2\pi}{T_{sr}} = \frac{2\pi\cdot\gamma^3 c}{R}$ and the corresponding wavelength is

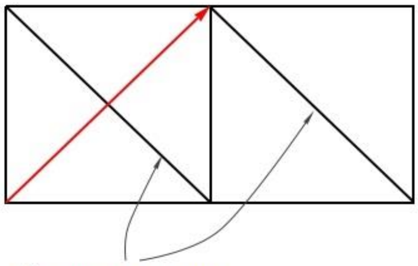
$$\lambda = \frac{2\pi c}{\omega_{sr}} = cT_{sr} \approx \frac{R}{\gamma^3} = \frac{R}{\left(\frac{E_e}{m_0 c^2}\right)^3} = \frac{m_0 c^2}{eBc\gamma^2} \approx 0{,}392 \cdot 10^{-10}\, m = 0{,}392 \text{ Å}.$$

6) Let us calculate the optical path for the waves reflected from adjacent planes (see the figure):



$\Delta s = |AC| + |CD| - |AB| = 2\frac{d}{\sin\theta} - 2d \cdot \text{ctg}\theta \cdot \cos\theta = 2d \cdot \sin\theta$. Constructive interference (amplification) is observed when the optical path difference for these two waves equals an integer of wavelengths. Hence, the condition of diffraction maximum is: $2d \cdot \sin\theta = n\lambda$, where n is an integer (Bragg's law). The absolute value of n is called the *order* of diffraction maximum.

7) The diagram below shows the projection of a unit cell on the XY-plane. The red arrow is the vector perpendicular to the planes of set (110).



Atomic planes perpendicular to vector (110).

Using the diagram and the fact that the cell is a cube one evaluates the interplanar spacing d_x for the set of planes (110): $d_x = \frac{L}{\sqrt{2}} \approx 3{,}74$ Å. Then the diffraction angle for the maximum of first order (for reflex (110)) follows from Bragg's law: $\sin\theta = \frac{\lambda}{2d_x} = \frac{0{,}392}{2 \cdot 3{,}74} \approx 0{,}0524$. The angle between the incident and diffracted beams equals 2θ, i.e. $2\theta \approx \frac{\lambda\sqrt{2}}{L} \approx 0{,}105$ рад $\approx 6°$.

8) Since the net occupation of the niobium positions (3-8) must remain equal to 1 the occupation of the positions by Nb atoms is

$$a_3(\text{Nb}) = a_4(\text{Nb}) = a_5(\text{Nb}) = a_6(\text{Nb}) = a_7(\text{Nb}) = a_8(\text{Nb}) = 1 - \delta.$$

If N is the total amount of Sn atoms (and the net number of positions 1 and 2 as well) in the structure, the number of Nb atoms (and positions 3-8) equals 3N (according to the chemical formula). For a given δ the number of atoms Sn switching to Nb positions equals $\delta \cdot 3N$; the same amount of Nb atoms move to Sn positions. Thus, the occupation of positions 1 and 2 by Nb atoms equals $a_1(\text{Nb}) = a_2(\text{Nb}) = 3\delta$. Accordingly, $a_1(\text{Sn}) = a_2(\text{Sn}) = 1 - 3\delta$.

9) Since the sum in the formula for the structure factor is quite cumbersome, it is easier to compute it by separating contributions of six Nb positions (3-8) and two positions (1-2) of Sn. According to equation given in 9) the contribution to the reflex structure factor F of an arbitrary reflex (hkl) due to positions 3-8, which occupations a by atoms of a single element with atomic factor f are the same, is

$$F_{3-8} = a \cdot f \cdot \left[e^{2i\pi\left(\frac{h}{4}+\frac{l}{2}\right)} + e^{2i\pi\left(\frac{3h}{4}+\frac{l}{2}\right)} + e^{2i\pi\left(\frac{h}{2}+\frac{k}{4}\right)} + e^{2i\pi\left(\frac{h}{2}+\frac{3k}{4}\right)} + e^{2i\pi\left(\frac{k}{2}+\frac{l}{4}\right)} + e^{2i\pi\left(\frac{k}{2}+\frac{3l}{4}\right)}\right]$$
$$= a \cdot f \cdot \left[e^{i\pi l}\left(e^{i\pi\frac{h}{2}} + e^{i\pi\frac{3h}{2}}\right) + e^{i\pi h}\left(e^{i\pi\frac{k}{2}} + e^{i\pi\frac{3k}{2}}\right) + e^{i\pi k}\left(e^{i\pi\frac{l}{2}} + e^{i\pi\frac{3l}{2}}\right)\right]$$

(here coordinates of every position are used). The contribution of positions 1 and 2 is

$$F_{12} = a \cdot f \cdot \left[e^{2i\pi(0\cdot h+0\cdot k+0\cdot l)} + e^{2i\pi\left(\frac{h}{2}+\frac{k}{2}+\frac{l}{2}\right)}\right] = a \cdot f \cdot \left[1 + e^{i\pi(h+k+l)}\right].$$

For the reflex $(hkl) = (110)$ this gives:

$$F_{3-8}(110) = a \cdot f \cdot \left[e^{i\pi 0}\left(e^{i\pi\frac{1}{2}} + e^{i\pi\frac{3}{2}}\right) + e^{i\pi}\left(e^{i\pi\frac{1}{2}} + e^{i\pi\frac{3}{2}}\right) + e^{i\pi}(e^0 + e^0)\right] =$$
$$= a \cdot f \cdot [1 \cdot (i - i) - 1(i - i) - 1(1 + 1)] = -2af,$$

and

$$F_{12}(110) = a \cdot f \cdot \left[1 + e^{i\pi(1+1+0)}\right] = 2af.$$

Now let us compute F using occupations calculated in 8) by summing the contributions of all eight positions:

$$F(110) = 2f(\mathrm{Nb}) \cdot 3\delta + 2f(\mathrm{Sn}) \cdot (1 - 3\delta) - 2f(\mathrm{Nb}) \cdot (1 - \delta) - 2f(\mathrm{Sn}) \cdot \delta =$$
$$= 6f_{\mathrm{I}}\delta + 2f_{\mathrm{II}} - 6f_{\mathrm{II}}\delta - 2f_{\mathrm{I}} + 2f_{\mathrm{I}}\delta - 2f_{\mathrm{II}}\delta = 2(f_{\mathrm{II}} - f_{\mathrm{I}}) \cdot [1 - 4\delta].$$

10) The reflex intensity equals $I_{110} \approx |F(110)|^2 = 4|f_{\mathrm{II}} - f_{\mathrm{I}}|^2 \cdot [1 - 4\delta]^2$, it vanishes when $\delta = 0{,}25$. Notice, that it is for this particular value of δ all occupations of atoms of a given element are equal: for any position n occupations $a_n(\mathrm{Nb}) = 0{,}75$ and $a_n(\mathrm{Sn}) = 0{,}25$. No wonder, this corresponds to the chemical composition of the material (25% of tin and 75% of niobium) because average occupations must satisfy this property for any δ.

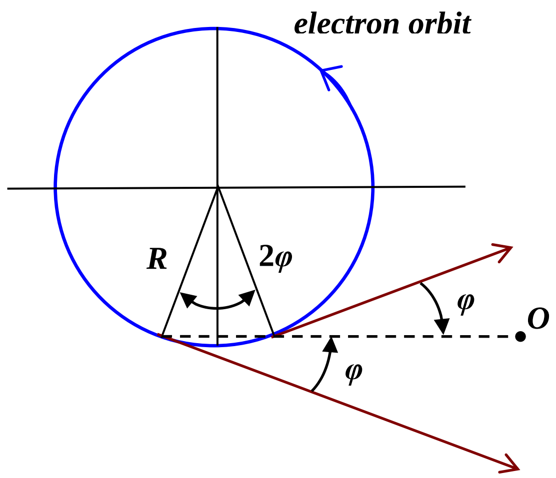
TABLE OF ANSWERS

№	**Answer**	**Maximum score**
1	$\gamma = \frac{E_e}{m_0 c^2} \approx 5000.$	1+1=2
	$\frac{c - v}{c} \approx \frac{1}{2\gamma^2} \approx 2 \cdot 10^{-6}$ %.	
2	$R \approx \frac{E_e}{ecB} = \frac{\gamma m_0 c}{eB} \approx 4{,}9$ m.	3 (equation) +1 (number)=4
3	$\Delta l \approx \frac{2R}{\gamma} \approx \frac{2E_e}{ec\gamma B} = \frac{2m_0 c}{eB}$ (any variant).	1
4	$T_{sr} \approx \frac{R}{c\gamma^3} \approx 13{,}06 \cdot 10^{-20}\ s$	3 (equation) +1 (number)=4
5	$\lambda \approx \frac{R}{\gamma^3} = \frac{R}{\left(\frac{E_e}{m_0 c^2}\right)^3} = \frac{m_0 c^2}{eBc\gamma^2} \approx 0{,}392 \cdot 10^{-10}\ m =$ 0,392 Å.	1(equation) +1 (number)=2
6	$2d \cdot \sin\theta = n\lambda$, where n is an integer.	5
7	$2\theta \approx \frac{\lambda\sqrt{2}}{L} \approx 0{,}105$ rad $\approx 6°$.	3 (equation) +1 (number)=4
8	$a_{3-8}(\mathrm{Nb}) = 1 - \delta;$	1+1,5+1,5=4
	$a_{1,2}(\mathrm{Nb}) = 3\delta;$	
	$a_{1,2}(\mathrm{Sn}) = 1 - 3\delta.$	
9	$F_{\mathrm{Nb}}(110) = -2af;$	1,5+1,5+3=6
	$F_{\mathrm{Sn}}(110) = 2af;$	
	$F(110) = 2(f_{\mathrm{II}} - f_{\mathrm{I}}) \cdot [1 - 4\delta].$	
10	$\delta = 0{,}25.$	3
Всего		**35**

Problem 2: RADIOGRAPHY

PROPOSED SOLUTION AND ANSWERS

1) According to the given equations $\gamma = \frac{E_e}{m_0 c^2} \approx \frac{2500}{0{,}5} = 5000$, i.e. $\gamma \gg 1$. Using the definition of the Lorentz factor one finds: $\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}$, whence $\frac{c-v}{c} \approx \frac{1}{2\gamma^2}$. Therefore, the speed of electrons in this machine is less than the speed of light by approximately 0,000002%.
2) In the field of bending magnet electrons are subjected to the Lorentz force which is perpendicular to particle velocity and does no work. Therefore, the absolute value of electron velocity (and the momentum) remains constant. The Lorentz factor remains constant as well and the equations of motion coincide with the corresponding equations of Newtonian mechanics in which mass is replaced with relativistic mass $m = \gamma m_0$. Hence, the curvature radius of electron path (called «Larmor radius») can be found from the equation $\frac{\gamma m_0 v^2}{R} = evB$. Since the electron speed is practically the same as the speed of light one obtains: $R \approx \frac{E_e}{ecB} = \frac{\gamma m_0 c}{eB}$, or $R[m] = \frac{E[GeV]\cdot 10^9 \cdot 1{,}6\cdot 10^{-19}\left[\frac{J}{eV}\right]}{3\cdot 10^8\left[\frac{m}{s}\right]\cdot 1{,}6\cdot 10^{-19}[C]\cdot B[T]} \approx 4{,}9$ m.
3) If the «observer» of X-ray beam resides at the orbital plane, the detected radiation is produced by electrons traveling along the arc subtending the angle equal to the beam aperture (the cone aperture), see the figure. The corresponding arc length $\Delta l = R \cdot 2\varphi \approx \frac{2R}{\gamma}$. Equivalent answers are $\Delta l \approx \frac{2E_e}{ec\gamma B}$ and $\Delta l \approx \frac{2m_0 c}{eB}$.



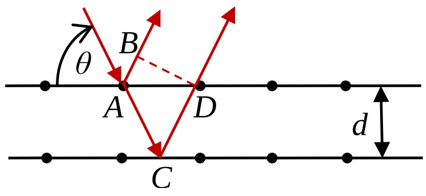
4) The time required for an electron to cover a distance Δl is $T = \frac{\Delta l}{v} \approx \frac{2R}{\gamma c}$, however, the duration of the flash is less due to the finiteness of the speed of light. To understand what is going on in detail, let us introduce the detector reference frame and consider two events: 1) emission of light by the electron at the beginning and 2) at the end of a flash. Let $t_1 = 0$ in the detector frame. Let the axis x be directed from the detector to the electron (since φ is small we can think of electron going along this axis during all the flash). Let electron coordinate (in electron frame) at $t_1 = 0$ be $x_1 = r$. The event 2 in electron frame then happens at $t_2 = T$ at the point $x_2 = r - 2Rsin(\varphi)$ (the electron approaches the detector). Now consider events: 3) detection of the beginning and 4) the end of flash by the detector. Obviously, the corresponding times are $t_3 = \frac{r}{c}$, $x_2 = r -$

$2Rsin(\varphi)$. Hence, $t_4 = \frac{r}{c} + T\left(1 - \frac{v}{c}\frac{sin\varphi}{\varphi}\right) \approx \frac{r}{c} + T\left(1 - \frac{v}{c}\right) + T\frac{\varphi^2}{6}$. Thus, the flash duration registered by the detector is $T_{sr} = t_4 - t_3 = \frac{2}{3\gamma^2}T = \frac{4R}{3c\gamma^3} \approx 17{,}4 \cdot 10^{-20}$ s.

5) According to the above analysis the corresponding characteristic frequency is $\omega_{sr} = \frac{2\pi}{T_{sr}} = \frac{3\pi\cdot\gamma^3 c}{2R}$ and the corresponding wavelength is

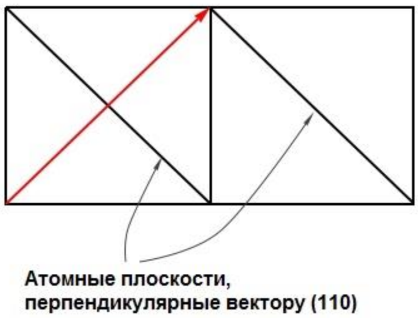
$$\lambda = \frac{2\pi c}{\omega_{sr}} = cT_{sr} \approx \frac{4R}{3\gamma^3} = \frac{4R}{3\left(\frac{Ee}{m_0c^2}\right)^3} = \frac{4m_0c^2}{3eBc\gamma^2} \approx 0{,}523 \cdot 10^{-10}\, m = 0{,}523 \text{ Å}.$$

6) Let us calculate the optical path for the waves reflected from adjacent planes (see the figure):



$\Delta s = |AC| + |CD| - |AB| = 2\frac{d}{\sin\theta} - 2d \cdot \text{ctg}\theta \cdot \cos\theta = 2d \cdot \sin\theta$. Constructive interference (amplification) is observed when the optical path difference for these two waves equals an integer of wavelengths. Hence, the condition of diffraction maximum is: $2d \cdot \sin\theta = n\lambda$, where n is an integer (Bragg's law). The absolute value of n is called the *order* of diffraction maximum.

7) The diagram below shows the projection of a unit cell on the *XY*-plane. The red arrow is the vector perpendicular to the planes of set (110).



Atomic planes perpendicular to vector (110).

Using the diagram and the fact that the cell is a cube one evaluates the interplanar spacing d_x for the set of planes (110): $d_x = \frac{L}{\sqrt{2}} \approx 3{,}74$ Å. Then the diffraction angle for the maximum of first order (for reflex (110)) follows from Bragg's law: $\sin\theta = \frac{\lambda}{2d_x} = \frac{0{,}523}{2\cdot 3{,}74} \approx 0{,}07$. The angle between the incident and diffracted beams equals 2θ, i.e. $2\theta \approx \frac{\lambda\sqrt{2}}{L} \approx 0{,}14$ рад $\approx 8°$.

8) Since the net occupation of the niobium positions (3-8) must remain equal to 1 the occupation of the positions by Nb atoms is

$$a_3(\text{Nb}) = a_4(\text{Nb}) = a_5(\text{Nb}) = a_6(\text{Nb}) = a_7(\text{Nb}) = a_8(\text{Nb}) = 1 - \delta.$$

If N is the total amount of Sn atoms (and the net number of positions 1 and 2 as well) in the structure, the number of Nb atoms (and positions 3-8) equals 3N (according to the chemical formula). For a given δ the number of atoms Sn switching to Nb positions equals $\delta\cdot 3N$; the same amount of Nb atoms move to Sn positions. Thus, the occupation of positions 1 and 2 by Nb atoms equals $a_1(\text{Nb}) = a_2(\text{Nb}) = 3\delta$. Accordingly, $a_1(\text{Sn}) = a_2(\text{Sn}) = 1 - 3\delta$.

9) Since the sum in the formula for the structure factor is quite cumbersome, it is easier to compute it by separating contributions of six Nb positions (3-8) and two positions (1-2) of Sn. According to equation given in 9) the contribution to the reflex structure factor F of an arbitrary reflex

(hkl) due to positions 3-8, which occupations a by atoms of a single element with atomic factor f are the same, is

$$F_{3-8} = a \cdot f \cdot \left[e^{2i\pi\left(\frac{h}{4}+\frac{l}{2}\right)} + e^{2i\pi\left(\frac{3h}{4}+\frac{l}{2}\right)} + e^{2i\pi\left(\frac{h}{2}+\frac{k}{4}\right)} + e^{2i\pi\left(\frac{h}{2}+\frac{3k}{4}\right)} + e^{2i\pi\left(\frac{k}{2}+\frac{l}{4}\right)} + e^{2i\pi\left(\frac{k}{2}+\frac{3l}{4}\right)}\right]$$
$$= a \cdot f \cdot \left[e^{i\pi l}\left(e^{i\pi\frac{h}{2}} + e^{i\pi\frac{3h}{2}}\right) + e^{i\pi h}\left(e^{i\pi\frac{k}{2}} + e^{i\pi\frac{3k}{2}}\right) + e^{i\pi k}\left(e^{i\pi\frac{l}{2}} + e^{i\pi\frac{3l}{2}}\right)\right]$$

(here coordinates of every position are used). The contribution of positions 1 and 2 is

$$F_{12} = a \cdot f \cdot \left[e^{2i\pi(0\cdot h+0\cdot k+0\cdot l)} + e^{2i\pi\left(\frac{h}{2}+\frac{k}{2}+\frac{l}{2}\right)}\right] = a \cdot f \cdot \left[1 + e^{i\pi(h+k+l)}\right].$$

For the reflex $(hkl) = (110)$ this gives:

$$F_{3-8}(110) = a \cdot f \cdot \left[e^{i\pi 0}\left(e^{i\pi\frac{1}{2}} + e^{i\pi\frac{3}{2}}\right) + e^{i\pi}\left(e^{i\pi\frac{1}{2}} + e^{i\pi\frac{3}{2}}\right) + e^{i\pi}(e^0 + e^0)\right] =$$
$$= a \cdot f \cdot [1 \cdot (i - i) - 1(i - i) - 1(1 + 1)] = -2af,$$

and

$$F_{12}(110) = a \cdot f \cdot \left[1 + e^{i\pi(1+1+0)}\right] = 2af.$$

Now let us compute F using occupations calculated in 8) by summing the contributions of all eight positions:

$$F(110) = 2f(\mathrm{Nb}) \cdot 3\delta + 2f(\mathrm{Sn}) \cdot (1 - 3\delta) - 2f(\mathrm{Nb}) \cdot (1 - \delta) - 2f(\mathrm{Sn}) \cdot \delta =$$
$$= 6f_{\mathrm{I}}\delta + 2f_{\mathrm{II}} - 6f_{\mathrm{II}}\delta - 2f_{\mathrm{I}} + 2f_{\mathrm{I}}\delta - 2f_{\mathrm{II}}\delta = 2(f_{\mathrm{II}} - f_{\mathrm{I}}) \cdot [1 - 4\delta].$$

10) The reflex intensity equals $I_{110} \approx |F(110)|^2 = 4|f_{\mathrm{II}} - f_{\mathrm{I}}|^2 \cdot [1 - 4\delta]^2$, it vanishes when $\delta = 0{,}25$. Notice, that it is for this particular value of δ all occupations of atoms of a given element are equal: for any position n occupations $a_n(\mathrm{Nb}) = 0{,}75$ and $a_n(\mathrm{Sn}) = 0{,}25$. No wonder, this corresponds to the chemical composition of the material (25% of tin and 75% of niobium) because <u>average</u> occupations must satisfy this property for any δ.

TABLE OF ANSWERS

№	**Answer**	**Maximum score**
1	$\gamma = \frac{E_e}{m_0 c^2} \approx 5000.$	0,5+0,5=1
	$\frac{c - v}{c} \approx \frac{1}{2\gamma^2} \approx 2 \cdot 10^{-6}$ %.	
2	$R \approx \frac{E_e}{ecB} = \frac{\gamma m_0 c}{eB} \approx 4{,}9$ m.	2 (equation) +0,5 (number)=2,5
3	$\Delta l \approx \frac{2R}{\gamma} \approx \frac{2E_e}{ec\gamma B} = \frac{2m_0 c}{eB}$ (any variant).	1
4	$T_{sr} \approx \frac{4R}{3c\gamma^3} \approx 17{,}4 \cdot 10^{-20}\ s$	2 (equation) +0,5 (number)=2,5
5	$\lambda \approx \frac{4R}{3\gamma^3} = \frac{4R}{3\left(\frac{E_e}{m_0 c^2}\right)^3} = \frac{4m_0 c^2}{3eBc\gamma^2} \approx 0{,}523 \cdot 10^{-10}\ m =$ 0,523 Å.	1(equation) +0,5 (number)=1,5
6	$2d \cdot \sin\theta = n\lambda$, where n is an integer.	2
7	$2\theta \approx \frac{\lambda\sqrt{2}}{L} \approx 0{,}14$ rad $\approx 8°$.	1 (equation) +0,5 (number)=1,5
8	$a_{3-8}(\mathrm{Nb}) = 1 - \delta;$	1+1+1=3
	$a_{1,2}(\mathrm{Nb}) = 3\delta;$	
	$a_{1,2}(\mathrm{Sn}) = 1 - 3\delta.$	
9	$F_{\mathrm{Nb}}(110) = -2af;$	1+1+1,5=3,5
	$F_{\mathrm{Sn}}(110) = 2af;$	
	$F(110) = 2(f_{\mathrm{II}} - f_{\mathrm{I}}) \cdot [1 - 4\delta].$	
10	$\delta = 0{,}25.$	1,5
Всего		**20**

Neutrino

Neutrino is one of the most peculiar elementary particles. It has no electric charge and does not participate in the strong interactions (which are responsible for stability of atomic nuclei). Physicists use the word «flavor» to specify a neutrino type. There are three neutrino flavors known to date: electron neutrino ν_e, muon neutrino ν_μ, and tau-neutrino ν_τ. A neutrino of each flavor has its antiparticle (antineutrino). The symbols used for the latter are the same as for neutrinos but with an upper bar: $\bar{\nu}_e$, $\bar{\nu}_\mu$, and $\bar{\nu}_\tau$.

Neutrino participates only in the weak interactions, the most famous process mediated by the weak interaction is β-decay. In this process a single neutron in atomic nucleus decays into a proton, an electron, and an electron antineutrino: $n \to p + e + \bar{\nu}_e$ (however, it would be a mistake to think than neutron is composed of these particles, there is also a process $p \to n + e^+ + \nu_e$!).

A neutrino is always created together with its antineutrino or a charged antilepton (positron e^+ (electron antiparticle), antimuon μ^+, or antitau-lepton τ^+). An antineutrino, in turn, is always created together with its neutrino or the corresponding charged lepton.

Neutrinos are extremely lightweight, their masses are several orders of magnitude less than masses of other matter particles. The precise values of neutrino masses are still unknown. Due to their small masses all neutrinos participating in nuclear reactions are *ultrarelativistic*, i.e. their velocities are very close to the speed of light in vacuum. The energy of such a neutrino of mass m and momentum $\vec{p}$ is almost independent of its mass:

$$E = \sqrt{m^2c^4 + c^2\vec{p}^2} \approx c\,|\,\vec{p}\,|.$$

A neutrino, like many other elementary particles, has *spin*, i.e. the proper angular momentum, which is non-zero even in the neutrino rest frame. A specific feature of all detected neutrinos (antineutrinos) is the negative (positive) sign of its spin component projected on the direction of neutrino (antineutrino) momentum. Loosely speaking, a neutrino does not have a «mirror reflection». Other elementary matter particles can have both signs of the spin component. Physicists explain this fact by saying that neutrinos with other spin component either do not exist, or do not participate even in the weak interactions (so they cannot be detected).

Physical constants and data (can be used in any part of the problem)

- speed of light in vacuum $c \approx 3 \cdot 10^8$ m/s;
- gravitational constant $G \approx 6{,}7 \cdot 10^{-11}$ m^3/(kg·s^2);
- Planck constant $\hbar \approx 10^{-34}$ J·s;
- proton radius $r_p \approx 10^{-15}$ m;
- Avogadro constant $N_A \approx 6 \cdot 10^{23}$ mole^{-1};
- hydrogen molar mass $\mu \approx 2$ g/mole;
- Solar mass $M_C \approx 2 \cdot 10^{30}$ kg;
- Solar radius $r_C \approx 7 \cdot 10^8$ m;
- mean radius of Earth's orbit $a \approx 1{,}5 \cdot 10^{11}$ m;
- eccentricity of Earth's orbit $\varepsilon \approx 0{,}017$;
- radius of «active» solar core where nuclear fusion proceeds and neutrinos are created $r_a \approx 1{,}2 \cdot 10^8$ m;
- range of electron density n_e inside the Sun from the active core to outer layers: from $5{,}9 \cdot 10^{31}$ m^{-3} to 10^{29} m^{-3};
- parsec (pc), an astronomical unit of length, 1 pc $\approx 3{,}2$ light year $\approx 3 \cdot 10^{16}$ m.

- electronvolt (eV) is the unit of energy equal to the work done by electrostatic force moving a single electron across potential difference of 1 V.

Part I: neutrino masses and oscillations.

The Nobel Prize of 2015 was awarded for the «discovery of neutrino oscillations indicating that neutrinos are massive». Neutrino oscillations is a process of interconversion of neutrino flavors. According to modern theoretical models the possibility of neutrino oscillations is indeed closely related to their masses (massless neutrinos cannot oscillate).

It should be noted that neutrinos like other elementary particles are not some «immutable» entities, rather they are *quanta* of a neutrino field (similarly to photons which are quanta of electromagnetic field). Therefore, in different physical situations they can appear in the states with different properties. For instance, neutrino state of a *certain flavor* (a state in which neutrino is created or annihilated in nuclear reactions) does not coincide with neutrino state of a *certain mass*.

To be specific, consider oscillation $\nu_e \leftrightarrow \nu_\mu$ (i.e. we neglect the third neutrino flavor). An intensive flow of neutrinos can be regarded as «almost classical» radiation of a given wavelength (here the analogy with electromagnetic wave, an «almost classical» flow of a large number of photons, applies again).

The existence of several neutrino states can be described by introducing a «polarization»: $\vec{u}(t,\vec{r}) = \vec{u}_1 \cos(\omega_1 t - \vec{k}\vec{r}) + \vec{u}_2 \cos(\omega_2 t - \vec{k}\vec{r})$. The quotation mark indicates that this polarization is not a polarization in the «regular» space, this is polarization in the «space of neutrino states» although for our purposes this is almost insignificant. The flux of neutrinos is proportional to $\vec{u}^{\,2}$. Notice that the frequency ω and wavevector k of the wave are related to the energy and momentum of neutrinos by the common quantum formulae: $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$, where $\hbar$ is *Planck constant*.

The difference of frequencies is due to difference in masses: for the same momentum $E_{1,2} = \sqrt{m_{1,2}^2 c^4 + c^2 \vec{p}^{\,2}} = \hbar\omega_{1,2}$. Obviously, an orthogonal «polarization» $\vec{u}_{1,2}$ corresponds to the neutrino state of certain mass $m_{1,2}$. Notice, that states of definite flavor (ν_e and ν_μ) correspond to another pair of orthogonal «polarizations» $\vec{u}_{e,\mu}$ **which do not coincide with** $\vec{u}_{1,2}$.

The polarizations $\vec{u}_{1,2}$ corresponding to certain masses and polarizations $\vec{u}_{e,\mu}$ corresponding to certain flavors are related as:

$$\begin{cases} \vec{u}_1 = \vec{u}_e \cos\vartheta - \vec{u}_\mu \sin\vartheta, \\ \vec{u}_2 = \vec{u}_e \sin\vartheta + \vec{u}_\mu \cos\vartheta. \end{cases}$$

Angle ϑ is called the «mixing angle» of ν_e and ν_μ. In this case ν_e and ν_μ indeed do not have «certain» masses and do not have a certain energy for a given momentum. For instance, a measurement of the energy of electron neutrino would «on average» yield the value $\langle E_e \rangle = E_1 \cos^2\vartheta + E_2 \sin^2\vartheta = E - \frac{\Delta E}{2}\cos 2\vartheta$, where $E \equiv \frac{E_1 + E_2}{2} \approx c\,|\,\vec{p}\,|$ is the mean neutrino energy and $\Delta E \equiv E_2 - E_1$. Such an outcome could be interpreted as being due to interaction of the states ν_e and ν_μ, where the interaction energy is $V_{e\mu} = \Delta E \sin\vartheta \cos\vartheta$.

The quantities introduced above can be expressed in terms of energy E and parameters $m \equiv \frac{m_1 + m_2}{2}$, $\Delta m \equiv m_2 - m_1$, and ϑ. It has been already mentioned that accurate values of these parameters are not known yet but to solve the problem it would suffice to adopt the following approximate values: $mc^2 = 4{,}0 \cdot 10^{-3}$ eV, $\Delta mc^2 = 3{,}0 \cdot 10^{-3}$ eV, and $\vartheta = 10°$.

1) Evaluate «mean» masses of electron and muon neutrinos for the given values of the parameters. The answer can be given either in kg or eV/c^2.

Now consider neutrinos radiated from some small region and propagating along *x*-axis. Let the energy of a neutrino created in this region be $E \approx 2$ MeV, all created neutrinos are electron neutrinos and have the same certain momentum (this means that neutrinos are created with different masses and, therefore, energies). Clearly, the neutrino wave propagating along *x*-axis is a mixture of neutrino waves corresponding to neutrinos of different masses (hence, a mixture of waves with certain frequencies at a given wavelength). The phase shift of the waves varies with distance *x*. Since the phase shift varies the contributions to the resulting wave due to electron and muon component would vary as well. Therefore, at any particular position *x* one would detect not only electron neutrinos but muon neutrinos as well. The intensities of the corresponding neutrino fluxes vary periodically in space. This phenomenon is called neutrino oscillations.

2) Determine the oscillation length, i.e. the period of spatial variation of the time averaged flux of muon neutrinos. The answer should be given as the formula and the numerical value.

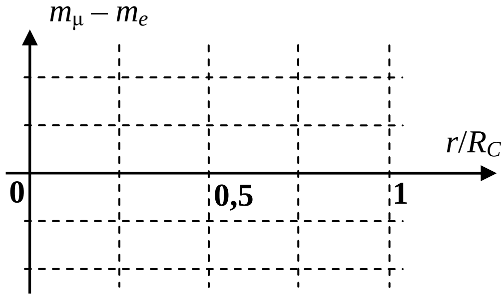
Part II: neutrinos and the Sun.

Oscillations described in Part I occur in vacuum. At first glance, it would be reasonable to assume that matter does not alter the picture significantly since neutrinos very weakly interact with any matter which density is much less than the density of atomic nucleus. There is a powerful neutrino source close to the Earth. It is the Sun. Nuclear reactions proceed in the central regions of the Sun suppling it with energy and creating neutrinos and antineutrinos, mostly, electron ones. However, the observed flux of electron neutrinos turned out to be only a half of the flux predicted from the Solar luminosity. Is it possible to explain this «deficit» by partial conversion of electron neutrinos to neutrinos of other flavors on their way from the Sun to the Earth (vacuum oscillations)?

3) Try to give a justified answer by using the data and the results from Part I of the problem. Do necessary calculations to support your judgement. In particular, it is reasonable to assume that energies of neutrinos created in nuclear reactions in the Sun are not very different from 2 MeV. Take into account the fact that a neutrino detector accumulates data for a long period of time, up to 2-3 months. The answer should be given as «+» (yes) or «–» (no).

A detailed analysis must take into account that the neutrinos travel a part of their path inside the solar substance. It turns out, absorption of the neutrinos by the substance does not change much the estimate for the flux of electron neutrinos but there is some additional circumstance which is quite essential. The solar substance contains a lot of electrons (see the problem data) and electron neutrinos interact with them much stronger than muon neutrinos do. Due to this fact the «mean» energy of electron neutrinos increases by δE, the quantity which is proportional to the electron density: $\delta E \approx 1{,}27 \cdot 10^{-12} \dfrac{n_e}{10^{31}\ \mathrm{m}^{-3}}$ eV. At the same time the «mean» energy of muon neutrinos and the energy of interaction between the states ν_e and ν_μ **remain practically the same**.

4) Evaluate «new» value of mixing angle $\tilde{\vartheta}$ (by taking into account the solar electrons). The answer should be the equation.
5) By how many percent can the solar electrons change the expected «deficit» of the electron neutrino flux? Evaluate in percent (%) the maximum increment of the flux of muon neutrinos due to oscillations (compared to the flux in the absence of matter).
6) Plot an approximate dependence (i.e. show only the main features) of the mass difference of ν_μ and ν_e as a function of the distance r traveled inside the Sun from the center outwards.



Part III: neutrino and supernova explosion.

After an «ordinary» star has exhausted its nuclear fuel the star cools down and its internal pressure cannot withstand gravitational compression anymore. As a result, the star is contracting until its substance undergoes transition to a new phase in which all electrons are «shared», all nuclei «float» in the electron «gas», and only the pressure of this gas halts further collapse.

The star can exist in this state for a long time; however, if the mass of its dense core gradually increases and exceeds $M_{\text{cr}} \approx 1{,}5 M_C$ the core becomes unstable and collapses. At the onset of collapse such a core usually has a radius about 10000 km with approximately equal numbers of protons and neutrons. Soon after the contraction starts the rate of electron-nuclear collisions becomes high, which results in *neutronization* of the star substance due to reaction of «inverse β-decay» ($p + e \to n + \nu_e$). Electron disappearance reduces the pressure of electron gas accelerating the neutronization even more. The whole process is essentially the tremendous explosion leaving in the aftermath a *neutron star* and an «outer envelope» flying outwards. Astronomers call such an explosion «supernova explosion» or simply «supernova». A neutron star is indeed composed mostly of tightly packed neutrons, so its density is approximately equal to the density of atomic nuclei.

7) Estimate an order of magnitude of the energy released due to compression of the stellar core from the initial radius to the neutron star. Calculate the numerical answer in Joules.

The released energy converts to kinetic energy of the star remnants (the flying outer layers and rotation of the neutron star) and to the energy of electromagnetic radiation and neutrinos. Supernova explosion is one of the most powerful source of neutrinos (sometimes it is called «neutrino bombs»), calculations show that more than half of the released energy converts to the energy of radiated neutrinos. Neutrinos are radiated both at the neutronization stage and after formation of the neutron star which is initially extremely hot and subsequently cools down mostly by radiating neutrinos. The neutronization and cooling take just several seconds. Notice that only electron neutrinos are created during the neutronization and neutrino-antineutrino pairs of various flavors are created during the cooling.

8) Estimate the number of neutrinos created by a supernova explosion assuming that the mass of the initial stellar core is approximately $1{,}5 M_C$ (the stellar substance does not «go away» with outer layers), the energy of radiated neutrinos is 80% of the released energy, and the mean energy of radiated neutrinos and antineutrinos is approximately 10 MeV. You could assume that neutron mass and radius are approximately the same as those of proton. Calculate the numerical values.
9) Supernova SN1987A exploded at the distance of $R \approx 50$ kpc from the Earth (in the Large Magellanic Cloud). What is the total number of neutrinos and antineutrinos passed through an Earth based detector of the cross-sectional area $S = 100$ m^2? Estimate the expected number of detected neutrinos and antineutrinos assuming that the detector on average registers $\alpha = 3 \cdot 10^{-14}$ % of neutrinos of any flavor in the corresponding energy range. Calculate the numerical values.

It is important that neutrino radiation is asymmetric with respect to the star magnetic axis due to «peculiar» neutrino behavior under mirror reflection: the power dI radiated in the infinitesimal solid angle $d\Omega = \sin\theta d\theta d\phi$ is $\frac{dI}{d\Omega} = \frac{I}{4\pi}[1+\kappa\cdot\cos\theta]$, where $\kappa \approx 10^{-2}$, θ is the angle of neutrino emission to the axis, and ϕ is the angle of rotation around the axis.

10) Estimate the speed gained by the star due to neutronization and cooling. Calculate the numerical answer (in km/s).

PROPOSED SOLUTION AND ANSWERS

Part I.

1. The particle mass in STR (rest mass) is determined from the relation between the particle energy and momentum. For ultrarelativistic particle $E = c|\vec{p}|\sqrt{1+\frac{m^2c^2}{\vec{p}^2}} \approx c|\vec{p}| + \frac{m^2c^3}{2|\vec{p}|}$. Therefore,

$$\langle E_e \rangle = E_1\cos^2\vartheta + E_2\sin^2\vartheta \approx c|\vec{p}| + \frac{(m_1^2\cos^2\vartheta + m_2^2\sin^2\vartheta)c^3}{2|\vec{p}|} = c|\vec{p}| + \frac{\langle m_e\rangle^2 c^3}{2|\vec{p}|}$$

and the «mean» mass of electron neutrino in vacuum

$$\langle m_e \rangle = \sqrt{m_1^2\cos^2\vartheta + m_2^2\sin^2\vartheta} = \sqrt{m^2 + \frac{(\Delta m)^2}{4} - m\,\Delta m\cos 2\vartheta}\,.$$

The numerical value $\langle m_e\rangle \approx 2{,}64\,\frac{\text{meV}}{c^2} \approx 4{,}69\cdot 10^{-39}$ kg.

Similarly, for muon neutrino, $\langle m_\mu \rangle = \sqrt{m^2 + \frac{(\Delta m)^2}{4} + m\,\Delta m\cos 2\vartheta}$ and $\langle m_\mu\rangle \approx 5{,}43\,\frac{\text{meV}}{c^2} \approx 9{,}66\cdot 10^{-39}$ kg.

2. An electron neutrino created at $t = 0$ and $x = 0$ is a superposition of states of certain masses: $\vec{u}_e = \vec{u}_1\cos\vartheta + \vec{u}_2\sin\vartheta$. Each state propagates at a distance x as a wave of certain frequency and wavevector, i.e.

$$\vec{u}(t,x) = \vec{u}_1\cos\vartheta\cos(\omega_1 t - kx) + \vec{u}_2\sin\vartheta\cos(\omega_2 t - kx)\,.$$

Substituting expressions for $\vec{u}_{1,2}$ in this formula one finds that at any point the wave contains two «flavor polarizations»:

$$\vec{u}(t,r) = \vec{u}_e[\cos^2\vartheta\cos(\omega_1 t - kx) + \sin^2\vartheta\cos(\omega_2 t - kx)] +$$
$$+\vec{u}_\mu\sin\vartheta\cos\vartheta[\cos(\omega_2 t - kx) - \cos(\omega_1 t - kx)].$$

The wave of muon neutrinos (the second term) equals to the expression

$$-\sin 2\vartheta\sin\left[\frac{\omega_2-\omega_1}{2}t\right]\sin\left[\frac{\omega_2+\omega_1}{2}t - kx\right],$$

corresponding to a propagating wave of the «mean» frequency $\frac{\omega_2+\omega_1}{2}$ and «amplitude» modulated by the harmonic factor of the half-difference frequency. Neutrinos travel a distance x from the point of emission for a time $t = \frac{x}{c}$ and the half-difference frequency is $\frac{\omega_2-\omega_1}{2} = \frac{E_2-E_1}{2\hbar} = \frac{\Delta E}{2\hbar}$. It is important to note that small corrections due to neutrino masses can be neglected when calculating the mean energy ($E \equiv \frac{E_1+E_2}{2} \approx c|\vec{p}|$) and must be taken into account when calculating the energy difference: $E_{1,2} = \sqrt{m_{1,2}^2c^4 + c^2\vec{p}^2} \approx \sqrt{m_{1,2}^2c^4 + E^2} \approx E + \frac{m_{1,2}^2c^4}{2E}$, hence $\Delta E = E_2 - E_1 \approx \frac{(m_2^2-m_1^2)c^4}{2E} = \frac{m\cdot\Delta m\cdot c^4}{E} = 6\cdot 10^{-12}$ eV. Obviously, $\Delta E \ll E$, so the «amplitude» factor can be regarded as almost constant during the oscillation period corresponding to the «mean» frequency. Therefore, the flux of muon neutrinos as a function of x obtained by averaging $\vec{u}^2$ over the period corresponding to the «mean» frequency is: $I_\mu(x) \approx I_0 \sin^2 2\vartheta \cdot \sin^2\left(\frac{m\cdot \Delta m \cdot c^3}{2E\hbar}x\right)$. Here I_0 is the initial flux of neutrinos. Therefore, the oscillation length corresponding to the spatial period of the second factor in I_μ equals $L = \frac{2\pi E\hbar}{m\cdot \Delta m \cdot c^3} \approx 2\cdot 10^5$ м = 200 km.

ANSWERS-1: the «mean» mass of ν_e equals $\langle m_e \rangle \approx 2{,}64\,\frac{\text{meV}}{c^2} \approx 4{,}69\cdot 10^{-39}$ kg, the «mean» mass of ν_μ equals $\langle m_\mu \rangle \approx 5{,}43\,\frac{\text{meV}}{c^2} \approx 9{,}66\cdot 10^{-39}$kg, and the oscillation length in vacuum is $L = \frac{2\pi E\hbar}{m\cdot \Delta m \cdot c^3} \approx 2\cdot 10^5$ m.

Note: *the contesters could estimate the «mean» masses somewhat differently, e.g. by following this line of argument:* «The particle mass is proportional to its energy at zero momentum, therefore, the «mean» mass of electron neutrino in vacuum is $\langle m_e \rangle = m_1 \cos^2\vartheta + m_2 \sin^2\vartheta = m - \frac{\Delta m}{2}\cos 2\vartheta$. Then $\langle m_e \rangle \approx 2{,}59\,\frac{\text{meV}}{c^2} \approx 4{,}61\cdot 10^{-39}$ kg. Similarly, for muon neutrino

$\langle m_\mu \rangle = m_1 \sin^2\vartheta + m_2 \cos^2\vartheta = m + \frac{\Delta m}{2}\cos 2\vartheta$, and $\langle m_\mu \rangle \approx 5{,}41\,\frac{\text{meV}}{c^2} \approx 9{,}62\cdot 10^{-39}$ kg».

This approach is not logical from the STR perspective since these «masses» are not invariant under Lorentz transformations, the correct mass must be defined via the invariant $E^2 - c^2|\vec{p}|^2$. *However, the numerical values turn out to be quite reasonable. That is why such an approach should be given a partial credit.*

Part II.

3. To estimate the impact of vacuum oscillations on the flux of electron neutrinos detected on the Earth, notice, that neutrinos entering the detector traveled various distances: they have been created at different points of active solar core of size $r_A >> L$. Besides, during the data accumulation the Earth changes its position with respect to the Sun (the distance between the Sun and the Earth varies by $\varepsilon a \approx 2{,}6\cdot 10^9\ \text{m} >> L$). Therefore, the flux must be averaged over variations of x which significantly exceed the oscillation length. Since $\left\langle \sin^2\left(\frac{\pi x}{L}\right)\right\rangle = \frac{1}{2}$, the flux of muon neutrinos corresponding to the decrement of the flux of electron neutrinos $I_\mu(x) \approx \frac{1}{2} I_0 \sin^2 2\vartheta \approx 0{,}058\cdot I_0$ which is less than 6 % of the initial value! Thus, vacuum oscillations cannot explain the significant decrease of the ν_e flux.

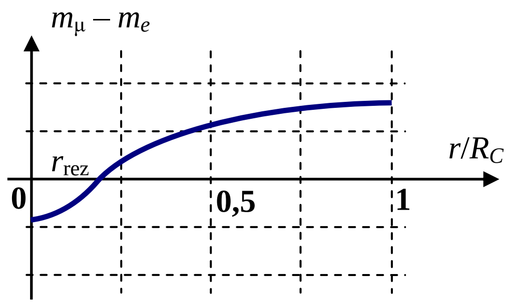
4. Proceeding to description of the oscillations in matter, notice, that although the energy of neutrino interaction with solar electrons is very small compared to the neutrino energy (even for the maximum electron density in the core $\delta E \approx 7{,}5\cdot 10^{-12}$ eV), it is of the same order as $\Delta E = 6\cdot 10^{-12}$ eV! Therefore, this interaction cannot have a significant impact on the neutrino propagation and/or the total flux but it can noticeably affect the oscillations.

Thus, taking into account the interaction of neutrinos with matter, the mean energy of electron neutrino becomes $\langle E_e \rangle = E - \frac{\Delta E}{2}\cos 2\vartheta + \delta E \equiv \tilde{E} - \frac{\Delta\tilde{E}}{2}\cos 2\tilde{\vartheta}$, where the «tilde» over a quantity corresponds to that one in matter. The mean energy of muon neutrinos and the energy of interaction of electron and muon neutrinos remain unaltered: $\langle E_\mu \rangle = E + \frac{\Delta E}{2}\cos 2\vartheta \equiv \tilde{E} + \frac{\Delta\tilde{E}}{2}\cos 2\tilde{\vartheta}$ and

$V_{e\mu} = \frac{\Delta E}{2}\sin 2\vartheta \equiv \frac{\Delta\tilde{E}}{2}\sin 2\tilde{\vartheta}$. Using the first two equations one finds that $\tilde{E} = E + \frac{\delta E}{2}$ and $\Delta\tilde{E}\cos 2\tilde{\vartheta} = \Delta E\cos 2\vartheta - \delta E$. Combing this and the third equation yields the remaining parameters: $\Delta\tilde{E} = \pm\sqrt{(\Delta E)^2 + (\delta E)^2 - 2\delta E\cdot\Delta E\cos 2\vartheta}$ and $\mathrm{tg}2\tilde{\vartheta} = \frac{\mathrm{tg}2\vartheta}{1-\delta E/(\Delta E\cos 2\vartheta)}$ (the minus sign in the expression for $\Delta\tilde{E}$ corresponds to $\delta E > \Delta E\cos 2\vartheta$).

5. First of all, let us look at the new value of the mixing angle: if $\delta E = \delta E_{rez} \equiv \Delta E\cos 2\vartheta$, the angle turns out to be $\vartheta_{rez} = \pi/4 = 45°$! This situation is called «maximal mixing» because the fractions of muon and electron neutrinos in the flow change periodically in antiphase between 0 и 100%. The phenomenon of sharp amplification of the oscillations in matter is called «Mikheyev–Smirnov–Wolfenstein effect» or MSW-effect. The condition of MSW-effect for neutrino with our parameters reads $1{,}27\cdot 10^{-12}\frac{n_e}{10^{31}\ \mathrm{m}^{-3}} \approx 6\cdot 10^{-12}\cdot\cos 20°$, i.e. $n_e = n_{rez} \approx 4{,}44\cdot 10^{31}\ \mathrm{m}^{-3}$. Notice, that the mixing angle away from the resonance remains small (the oscillations in these regions do not significantly affect the flux). One can also see that there is a layer outside the active solar core where electron density takes the «resonance» value. The width of the region in which the density differs from the resonant one by less than 1% is $\Delta r \approx 2\frac{R_C}{n_e(0)-n_e(R_C)}\cdot 0{,}01\cdot n_{rez} \approx 10^7$ m, which exceeds the oscillation length by more than order of magnitude. Therefore, a neutrino passing through this layer undergoes many oscillation cycles in which the fraction of muon neutrinos changes harmonically from 0% to 100%. The initial phase of the waves from different points in the solar core varies in a wide range. When evaluating the flux of ν_μ at the exit of resonance region one should average over these variations as well. Hence, the flux of muon neutrinos can be as large as 50% of the total flux (compare to 6% in vacuum). Thus, the solar matter increases the flux of ν_μ 8.5 times, or approximately by 750%!

6. Another interesting observation concerns neutrino masses. It follows from the equations for $\Delta\tilde{E}$ and $\tilde{\vartheta}$ that the difference of energies of muon and electron neutrinos with the same momentum $\langle E_\mu\rangle - \langle E_e\rangle = \Delta\tilde{E}\cdot\cos\tilde{\vartheta}$ becomes negative for $\delta E > \Delta E\cos 2\vartheta$ (i.e. when the electron density is higher than the resonant value)! Thus, in the solar core below the resonant layer the mass of ν_μ is less than the mass of ν_e. Outside the resonant layer $\delta E < \Delta E\cos 2\vartheta$ and the mass of ν_μ exceeds the mass of ν_e. Obviously, the mass difference tends to its «vacuum» value when the density is small. Therefore, the plot looks approximately as:



Note: An accurate calculation yields the same result. Using the expression for mean energy (see 1)) $E \equiv \frac{E_1+E_2}{2} \approx c\,|\vec{p}\,| + \frac{(m_1^2+m_2^2)c^4}{4E}$, one obtains for parameters $m,\Delta m$ (and similarly for $\tilde{m},\Delta\tilde{m}$):

$$\left.\begin{cases} E \approx c\,|\vec{p}| + \dfrac{[4m^2 + (\Delta m)^2]c^4}{8E}, \\ \tilde{E} = E + \dfrac{\delta E}{2} \approx c\,|\vec{p}| + \dfrac{[4\tilde{m}^2 + (\Delta\tilde{m})^2]c^4}{8E} \end{cases}\right\} \Rightarrow 4\tilde{m}^2 + (\Delta\tilde{m})^2 = 4m^2 + (\Delta m)^2 + \frac{4E\,\delta E}{c^4} \equiv 2M_1^2$$

(the quantity M_1 is introduced to simplify the calculation). Similarly, using the expression for $\Delta E \equiv E_2 - E_1$ (see 1)) and the formula for $\Delta\tilde{E}$ derived in 4), one obtains:

$$\left.\begin{cases} \Delta E \approx \dfrac{m\,\Delta m\,c^4}{E} \\ \Delta\tilde{E} \approx \dfrac{\tilde{m}\,\Delta\tilde{m}\,c^4}{E} \end{cases}\right\} \Rightarrow \tilde{m}\cdot\Delta\tilde{m} = \pm\sqrt{m^2(\Delta m)^2\sin^2 2\vartheta + [m\,\Delta m\cos 2\vartheta - E\,\delta E/c^4]^2} \equiv \pm\frac{M_2^2}{2}$$

(here M_2 is another shorthand notation).

Solving the set of these two equations allows one to determine $\tilde{m}$ and $\Delta\tilde{m}$. Mass is positive, therefore, $2\tilde{m} > \Delta\tilde{m}$ and

$$\begin{cases} \tilde{m} = \dfrac{1}{2}\sqrt{M_1^2 + \sqrt{M_1^4 - M_2^4}}, \\ \Delta\tilde{m} = \sqrt{M_1^2 - \sqrt{M_1^4 - M_2^4}}. \end{cases}$$

The quantity M_2 reaches its minimum at the resonant density (the second term under the radical in the expression for M_2 vanishes) while M_1 changes monotonically (slowly decreases with density since δE is proportional to the electron density and other terms in the expression for M_1 remain constant). One can see that $\Delta\tilde{m}$ decreases when approaching the resonant density. Besides, $\cos 2\tilde{\vartheta} = 0$ at the resonance. Therefore, the difference of masses of ν_μ and ν_e vanishes. It is obvious that $\cos 2\tilde{\vartheta}$ switches sign when crossing the resonant layer. When the density exceeds the resonant value the sign is negative, i.e. electron neutrino «on average» is heavier than the muon neutrino. When the density is below the resonant value, ν_e is lighter than ν_μ and the masses tend to their vacuum values when the density decreases. Clearly, the accurate analysis leads to the same conclusion as the proposed solution although the information obtained and the amount of calculations have significantly grown. Since the difference of mean masses of ν_μ and ν_e in matter is expressed via $\tilde{m}$, $\Delta\tilde{m}$, and $\tilde{\vartheta}$ one could write down the explicit formula for the mass difference as a function of electron density (it turns out to be quite cumbersome).

ANSWERS-2: $\mathrm{tg}2\tilde{\vartheta} = \dfrac{\mathrm{tg}2\vartheta}{1 - \delta E/(\Delta E\cos 2\vartheta)}$, solar matter enhances the flux of ν_μ approximately by 750%, the mass difference of ν_μ and ν_e at small electron density corresponds to the vacuum value, decreases when approaching the resonant density, goes through zero and changes sign when passing through the resonance layer, and then increases as the density grows.

Part III

7. Firstly, it is necessary to estimate the energy scale of the phenomenon, i.e. to determine the energy released during the collapse. The «binding energy» of a gravitating sphere of mass M and radius r equals $W_g = -\dfrac{3}{5}\dfrac{GM^2}{r}$. To estimate the radius of neutron star let us equate its density to the density of nuclear matter equal to the ratio of proton mass to its volume. The mass of N_A (1 mole) of protons approximately equals one half of molar mass of molecular hydrogen (μ = 2 г/моль), hence $\dfrac{M}{r^3} \approx \dfrac{\mu}{2N_A r_p^3}$ and $r \approx r_p\left(\dfrac{2N_A M}{\mu}\right)^{1/3}$. For a neutron star of mass $1{,}5M_C$ one obtains $r \approx 12$ km. The initial radius of the stellar core is greater by three orders of magnitude, so the released energy is $W^{(0)} \approx \dfrac{3}{5}\dfrac{GM^2}{r} \approx 3\cdot 10^{46}$ J.

Actually, this number is an overestimation because it has been assumed that the core mass does not change during the collapse. However, according to STR a decrease of energy is equivalent to the decrease of mass, i.e. one should actually write $W = \frac{3}{5}\frac{G(M - W/c^2)^2}{r}$. Since $Mc^2 \approx 2{,}7\cdot 10^{47}$ J $\approx 9W^{(0)}$, this effect is not too small. Solving the quadratic equation yields: $W = W^{(0)}\frac{1+2\beta-\sqrt{1+4\beta}}{2\beta^2}$, where $\beta \equiv \frac{W^{(0)}}{Mc^2} \approx \frac{1}{9}$. So, the better estimate of the released energy gives $W \approx 0{,}825W^{(0)} \approx 2{,}475\cdot 10^{46}$ J. In any case, $W \sim 10^{46}$ J by the order of magnitude.

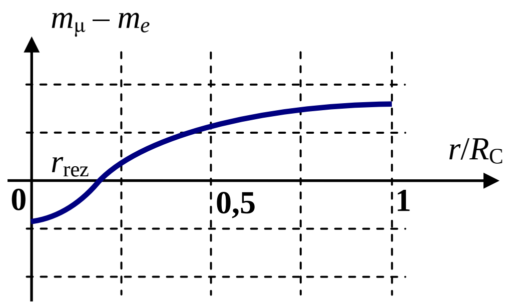
8. According to the problem statement the total energy carried away by neutrinos and antineutrinos $E_\nu \approx 0{,}8W \approx 2\cdot 10^{46}$ J. Since the average energy of emitted neutrinos and antineutrinos equals $\langle E_1\rangle \approx 10$ MeV, their total number is $N \approx \frac{E_\nu}{\langle E_1\rangle} \approx 1{,}25\cdot 10^{58}$. The number of proton-electron pairs in the initial stellar core is $N_{pe} \approx \frac{M}{2m_p} \approx \frac{N_A M}{\mu} \approx 0{,}09\cdot 10^{58}$, while in the final state almost only neutrons are left. Therefore, the number of neutrinos emitted during neutronization is $0{,}09\cdot 10^{58}$, and the rest are emitted during the cooling. Thus, the total number of emitted antineutrinos and neutrinos is: $N_{\bar\nu} \approx \left(\frac{1{,}25-0{,}09}{2}\right)\cdot 10^{58} = 5{,}8\cdot 10^{57}$ and $N_\nu \approx \left(0{,}09+\frac{1{,}25-0{,}09}{2}\right)\cdot 10^{58} = 6{,}7\cdot 10^{57}$.

9. The number of neutrinos and antineutrinos from SN1987A passing through the detector on Earth is proportional to the ratio of the detector cross-section to the area of a sphere of radius $R \approx 50$ kpc: $N^{(\mathrm{det})} \approx \frac{S}{4\pi R^2}N \approx 3{,}5\cdot 10^{-42}N$. Respectively, $N_{\bar\nu}^{(\mathrm{det})} \approx 2\cdot 10^{16}$ and $N_\nu^{(\mathrm{det})} \approx 2{,}35\cdot 10^{16}$. Since only the fraction $\alpha = 3\cdot 10^{-14}$ % is detected, the expected number of detected antineutrinos and neutrinos is estimated to be $N_{\bar\nu}^{(reg)} = \alpha N_{\bar\nu}^{(\mathrm{det})} \approx 6$ and $N_\nu^{(reg)} = \alpha N_\nu^{(\mathrm{det})} \approx 7$.

10. Since all emitted neutrinos and antineutrinos are ultrarelativistic, the momentum carried by them per small solid angle is $\frac{dI}{c}$. Therefore, the component of the momentum carried by the radiation projected on the magnetic axis z of the star is: $p_z = \frac{1}{c}\int dt\int_0^\pi \frac{dI}{d\Omega}\cos\theta d\Omega = \frac{\kappa}{4\pi c}\int Idt\int_0^\pi \cos^2\theta d\Omega$. Since $\int Idt = E_\nu$ and $\int \cos^2\theta d\Omega = 2\pi\int_0^\pi \sin\theta\cos^2\theta d\theta = \frac{4\pi}{3}$, one obtains $p_z = \frac{\kappa E_\nu}{3c}$. The recoil velocity is opposite to the magnetic axis and its value $V = \frac{p_z}{\tilde M}$, where $\tilde M$ is the mass of the created neutron star. Taking into account the mass defect $\tilde M \approx M - \frac{W}{c^2}$, one finally obtains $V = \frac{\kappa E_\nu}{3(Mc^2 - W)}c \approx 80$ km/s. This speed is already noticeable against the background motion of stars in galaxies (a typical velocity of this motion is several hundred km/s).

ANSWERS-3: the total number of neutrinos and antineutrinos emitted by supernova is $N_\nu \approx 6{,}7\cdot 10^{57}$ and $N_{\bar\nu} \approx 5{,}8\cdot 10^{57}$; the number of neutrinos and antineutrinos passing through the detector is $N_\nu^{(\mathrm{det})} \approx 2{,}35\cdot 10^{16}$ and $N_{\bar\nu}^{(\mathrm{det})} \approx 2\cdot 10^{16}$; and the expected number of detected neutrinos and antineutrinos is $N_\nu^{(reg)} \approx 7$ and $N_{\bar\nu}^{(reg)} \approx 6$. The recoil speed of neutron star is $V \approx 80$ km/s.

TABLE OF ANSWERS

№	Answer	Maximum score
1	$2{,}6\dfrac{\text{meV}}{c^2} \le \langle m_e \rangle \le 2{,}7\dfrac{\text{meV}}{c^2}$ or $4{,}6\cdot 10^{-36}$ kg $\le \langle m_e \rangle \le 4{,}8\cdot 10^{-36}$ kg. $5{,}3\dfrac{\text{meV}}{c^2} \le \langle m_\mu \rangle \le 5{,}5\dfrac{\text{meV}}{c^2}$ or $9{,}55\cdot 10^{-36}$ kg $\le < m_\mu > \le 9{,}75\cdot 10^{-36}$ kg.	3+3=**6** (if inside the «doubled» interval: 2+2=**4**)
2	Explicit expression for $\vec{u}(t,x)$ via «flavor polarization» or the amplitude of muon neutrino wave is written down. $L = \dfrac{2\pi E\hbar}{m\cdot \Delta m \cdot c^3} \approx 2\cdot 10^5$ m.	2+3 (2 for the equation + 1 for the value) = **5**
3	«–» (no) full credit is given if there is any reasonable justification supported by calculations.	**2**
4	The correct set of equations for $\tilde{E}$, $\Delta\tilde{E}$, and $\tilde{\vartheta}$ is written down. $\mathrm{tg}2\tilde{\vartheta} = \dfrac{\mathrm{tg}2\vartheta}{1-\delta E/(\Delta E\cos 2\vartheta)}$.	2 +3=**5**
5	The effect of «resonance» is discovered. Matter increases the flux of ν_μ by approximately 750%.	1+3=**4**
6	 (any plot with a finite negative value for $r < r_{rez}$ and a finite positive value for $r > r_{rez}$)	**3** (if it is not indicated that the r corresponds to $r = r_{rez}$: **–0,5** points)
7	Equation $W_g = -\dfrac{3}{5}\dfrac{GM^2}{r}$ is used (the factor 3/5 can be omitted). $W \sim 10^{46}$ J.	2 + 2 = **4** (factor 3/5 is absent or mass defect is not taken into account: **-0,5** points for each)
8	Neutrinos emitted at the neutronization stage are taken into account: about $0{,}09\cdot 10^{58}$. $N_\nu \approx 6{,}7\cdot 10^{57}$, $N_{\bar{\nu}} \approx 5{,}8\cdot 10^{57}$.	2 + 2 = **4** (mass defect is neglected: **3**, the error greater than 25% and less than 60%: **2**)
9	$N_\nu^{(\text{det})} \approx 2{,}35\cdot 10^{16}$ and $N_{\bar{\nu}}^{(\text{det})} \approx 2\cdot 10^{16}$. $N_\nu^{(reg)} \approx 7$ и $N_{\bar{\nu}}^{(reg)} \approx 6$.	1 + 1 = **2** (the error greater than 20% and less than 50%: **1**)
10	Integral expression for the momentum component is written down Momentum carried away, $p_z = \dfrac{kE_\nu}{3c}$. $V = \dfrac{\kappa E_\nu}{3(Mc^2 - W)}c \approx 80$ km/s.	1 + 2 + 2 = **5** (mass defect is neglected: **4**, the error is greater than 25% and less than 50%: **3**)
Total		**40**